The Mathematics of Simple Ultrasonic 2-Dimensional Sensing
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Introduction

Our company, Bitstream Technology, has been developing educational robotics products around radio-controlled (RC) cars. Our reasons for doing this are:

1. The RC car provides a cheap, ready-made platform
2. The RC car results in a fast autonomous vehicle that's fun to experiment with
3. The control electronics can be used on any RC vehicle, so the project is scalable

There are a number of microcontroller/electronics functions that we're developing to allow the vehicles to navigate autonomously in obstacle-rich environments. Of these functions, one of the most important is the collision avoidance sensor. We've made several different versions of this, consisting of various combinations of ultrasonic and optical sensors. The one described in this paper is an innovative and very useful enhancement of ultrasonic sensing boards.

The Ultrasonic Sensor System

The simple ultrasonic system has been around for a long time and is well understood. Basically, an ultrasonic transducer, which can be thought of as a high-frequency speaker, sends out a signal that will reflect back from obstacles. The ultrasonic sensor, which can be thought of as a high-frequency microphone, picks up these reflected waves and produces an electronic signal from them (Figure 1).

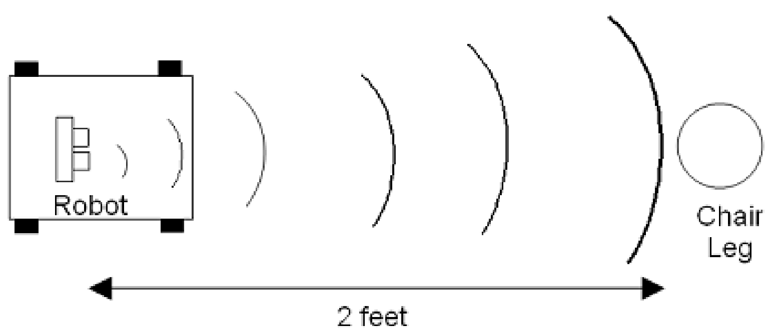

Figure 1 – Robotic vehicle using an ultrasonic sensor to detect an obstacle

The ultrasonic waveform transmitted through the air is an acoustic signal so it travels at the speed of sound, about 1100 feet per second. As a result, the robot can accurately determine the distance to the obstacle by just computing the difference between the time of the ultrasonic wave generation at the transmitter and the receipt of the reflected wave at the ultrasonic receiver. Figure 2 is an oscilloscope display of the transmitter input and the (amplified) receiver output for an obstacle just 5 inches away (having the obstacle so close allows both waveforms to be shown on the oscilloscope display)

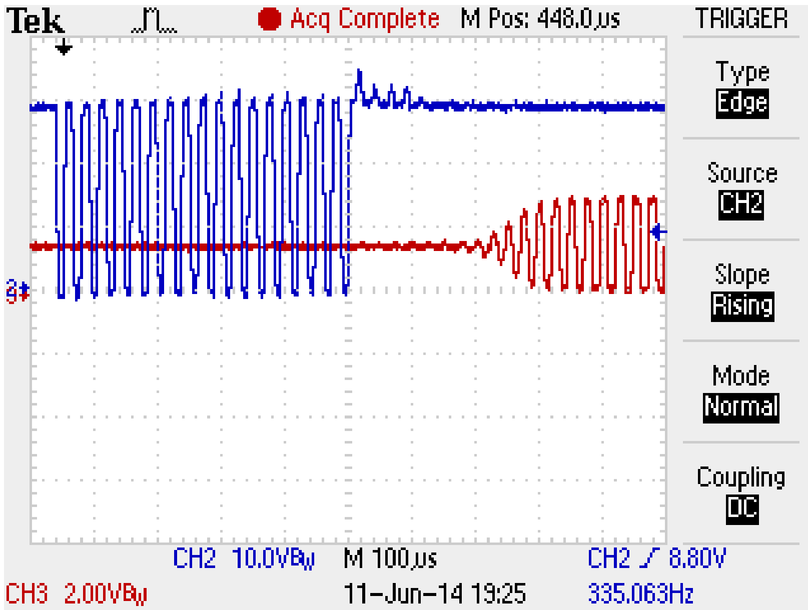

Figure 2 – 40 kilohertz ultrasonic transmitter output and reflected return (obstacle 5 inches away)

The Need for More Obstacle Information

Simple transmitter/receiver pairs like the one in Figure 1 are often used in slow-moving robots. When an obstacle is encountered, the response is often programmed so that the vehicle backs up and turns to one side or the other. But for higher-performance robots and autonomous vehicles, much better responses are possible if we just have a little more information.

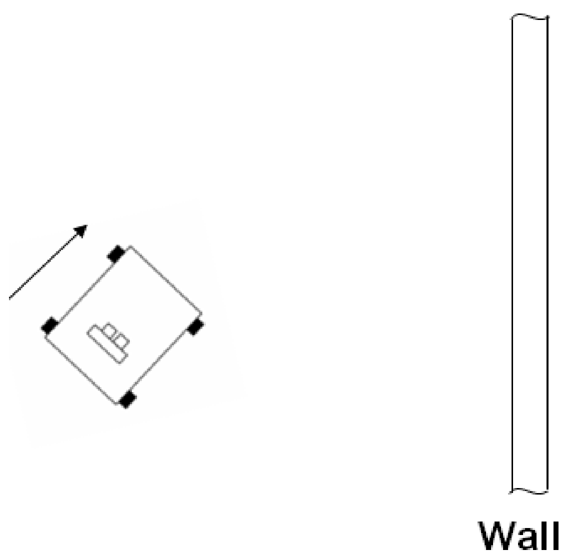

Figure 3 – Autonomous vehicle approaching wall on its right

Figure 3 illustrates a simple example. If the goal of the autonomous vehicle is to run around the course as fast as possible without running into anything, the correct maneuver is simply to turn the front wheels briefly to the left and keep going. Yet, the only information that our simple ultrasonic transmitter/receiver pair can give is that there's something in front of the vehicle at a distance, *D*. There's no additional information about whether the obstacle is to the left or to the right.

2-Dimensional Sensing Using Two Receivers and Some Math

We can modify the sensor in a way that provides significantly more information. This sensor will tell us not only that there is an obstacle ahead of us, but whether the obstacle is on the left or the right and how far to the left or right.

To do this we just add one ultrasonic receiver, as depicted in Figure 4, and some mathematics (algebra and geometry).

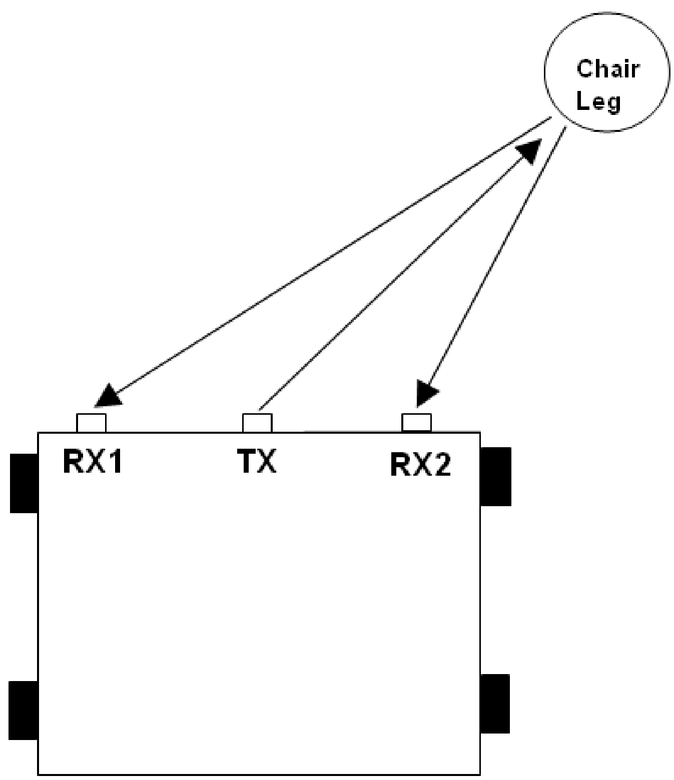

Figure 4 – 2-Dimensional ultrasonic sensor system

Before we talk about the mathematics, let's look at the sensing hardware. The sensor system sends ultrasonic energy from its transmitter (as usual). It propagates in all directions, but the only energy we're interested in is that energy that hits the chair leg, which is shown by the outgoing arrow in Figure 4. That energy is reflected back, again in all directions, but we're only interested in the energy that strikes one of the two receivers (RX1 or RX2) and that energy is depicted by the incoming arrows in Figure 4.

Remember from the earlier discussion that ultrasonic sensing is accomplished by measuring the time-of-flight (TOF) between transmission and receipt of the reflected ultrasonic energy. Observe in the example of Figure 4 that the path from transmitter, TX, to receiver, RX1, is longer than the path from the transmitter to the receiver, RX2.

The question is: if we know the TOF from transmitter-to-RX1 (call it *TOF1*) and the TOF from transmitter-to-RX2 (*TOF2*), is there an equation that we can use to compute the distance, y, and the distance, x, as defined in Figure 5? Note that y is the distance normal to the axis running through the ultrasonic transmitter and receivers and x is the distance along that axis.

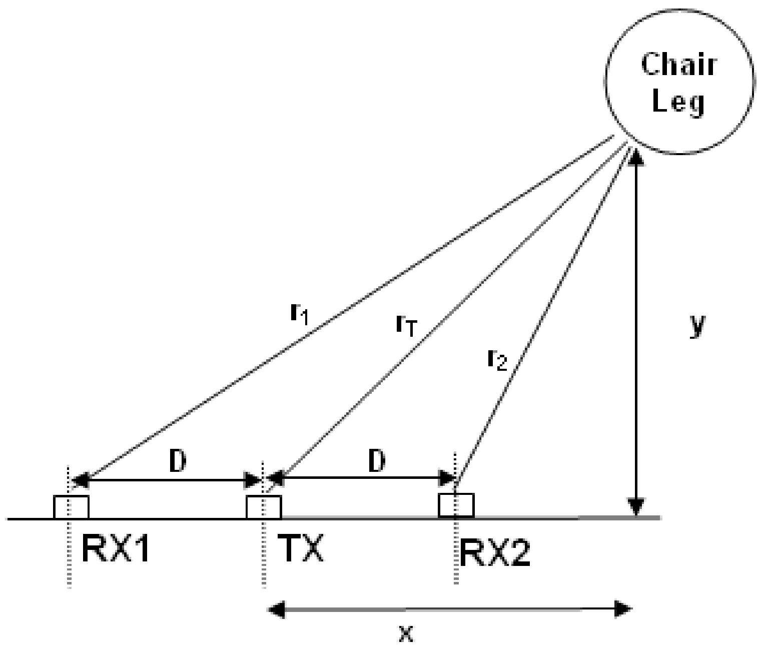

Figure 5 – Solving the *x,y* Computation Using Right Triangles

There are multiple ways to solve this problem but we will use methods that rely solely on algebra and geometry. As Figure 5 makes clear, we are dealing with three right triangles, all of which have a common side of length y and the other side either x-D, x, or x+D, where D is the spacing between the transmitter and each of the two receivers, RX1 and RX2. The hypotenuses of these three right triangles have length r_1, r_T, and r_2.

So what values do we know or can measure? We know D, the transmitter-to-receiver spacing, since it is part of the sensor circuit board design. Our robot will measure the times-of-flight, *TOF1* and *TOF2*. That's all we know. So how do we get x and y from that? A bunch of algebra and geometry, that's how.

Note that the hypotenuses and times-of-flight are related by the equations:

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$$TOF1 = \frac{r_T + r_1}{c} \quad \text{(Eq. 1)}$$

$$TOF2 = \frac{r_T + r_2}{c}. \quad \text{(Eq. 2)}$$

The speed of sound, *c*, at standard temperature and pressure is 1126 feet per second. The difference between these two times can be immediately written as:

$$TOF1 - TOF2 = \frac{r_1 - r_2}{c}. \quad \text{(Eq. 3)}$$

The sum of the two times-of-flight is given by:

$$TOF1 + TOF2 = \frac{r_1 + r_2 + 2r_T}{c}. \quad \text{(Eq. 4)}$$

For situations in which the ultrasonic receive sensors are close to the transmitter, relative to any of the distances being measured (that is, $D << rn$, where n is 1, 2, or T in Figure 5), the following approximation produces negligible error:

$$r_T \approx \frac{(r_1 + r_2)}{2}. \quad \text{(Eq. 5)}$$

Using this approximation, Eq. 4 becomes:

$$\frac{c}{2}(TOF1 + TOF2) = r_1 + r_2. \quad \text{(Eq. 6)}$$

By the Pythagorean theorem, the following three equations can be written:

$$x^2 + y^2 = r_T^{\ 2} \quad \text{(Eq. 7)}$$

$$(x + D)^2 + y^2 = r_1^{\ 2} \quad \text{(Eq. 8)}$$

$$(x - D)^2 + y^2 = r_2^{\ 2} \quad \text{(Eq. 9)}$$

Subtracting one from the other of these last two equations produces the result:

$$x = \frac{r_1^{\ 2} - r_2^{\ 2}}{4D} = \frac{c^2}{8D}(TOF1 - TOF2)(TOF1 + TOF2). \quad \text{(Eq. 10)}$$

Combining Eqs. 5, 6, and 7 gives:

$$x^2 + y^2 = \left(\frac{r_1 + r_2}{2}\right)^2 = \left[c\frac{TOF1 + TOF2}{4}\right]^2 \quad \text{(Eq. 11)}$$

and applying the result of Eq. 10 to Eq. 11 gives:

$$y = \sqrt{\frac{c^2}{16}(TOF1 + TOF2)^2 - \left(\frac{r_1^{\ 2} - r_2^{\ 2}}{4D}\right)^2}. \quad \text{(Eq. 12)}$$

or

$$y = \sqrt{\frac{c^2}{16}(TOF1 + TOF2)^2 - \left[\frac{c^2}{8D}(TOF1 - TOF2)(TOF1 + TOF2)\right]^2}. \quad \text{(Eq. 13)}$$

After some manipulation, this becomes:

$$y = \frac{c}{4}(TOF1 + TOF2)\sqrt{1 - \frac{c^2}{4}\left[\frac{(TOF1 - TOF2)}{D}\right]^2}\,. \qquad \text{(Eq. 14)}$$

Thus, computing just the sum and difference of the two times-of-flight allows an accurate estimate of the object's *x* and *y* displacement to the transmitter to be made.

Hardware/Software Implementation

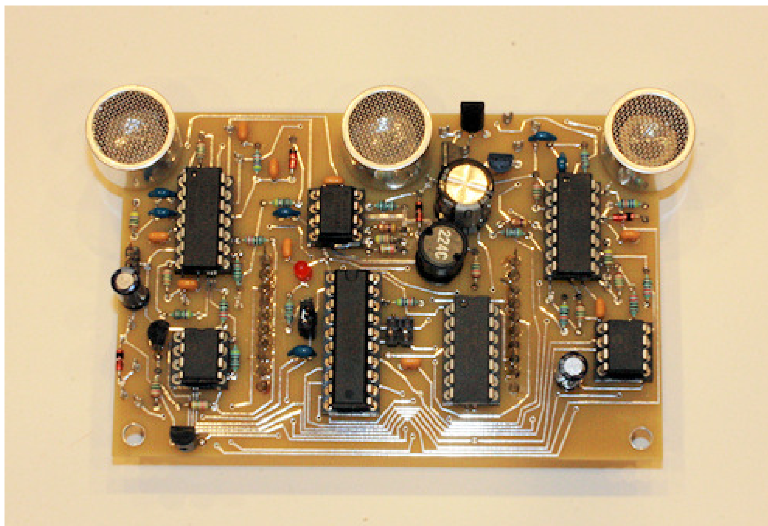

Figure 6 – The Bitstream Technology 2-dimensional ultrasonic sensor board

An ultrasonic transmitter and two receivers are incorporated into a 2-dimensional robotic sensor board (Figure 6) that Bitstream Technology has developed. The board includes a Texas Instruments MSP430 microcontroller that makes the computations explained in this paper. The microcontroller is preprogrammed, so that the user does not have to know how to make the floating-point calculations.

The output of the board is two voltages, which represent the distances *x* and *y* with scale factors of 0.2 volt per foot for *y* and 1 volt per foot for *x*. The resolution of the two outputs is approximately 1 inch for *y* and 0.3 inch for *x*.

The company's plan is to offer this board as part of a Kickstarter or Dragon Innovation project later this year. If you think you might be interested in supporting this project, please visit us at www.bitstreamtechnology.com.